**JAMES RUSE AGRICULTURAL HIGH SCHOOL**
**TERM 1 ASSESSMENT 1998**
**YEAR 12 3/4 UNIT**

**Time allowed: 85 minutes.**
All questions are to be attempted
Each question is to be handed in separately.

## Question 1 : START A NEW PAGE

(a) Evaluate exactly : $\sin^{-1}\left(\frac{-1}{2}\right)$

(b) Evaluate : $\lim_{x\to 0}\left(\frac{\sin 4x}{2x}\right)$

(c) Differentiate with respect to x :

(i) $y = 2\tan(x^3)$

(ii) $y = x\sin^{-1}x$

(d) Find the exact value of the gradient to the curve $y = \operatorname{cosec} x$ at $x = \frac{7\pi}{6}$ .

(e) Find $\frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right)$ in simplest terms.

## Question 2 : START A NEW PAGE

(a) Evaluate : $\int_{-4}^{4}\frac{6\,dx}{x^2+16}$

(b) Find : $\int\frac{2\,dx}{\sqrt{3-4x^2}}$

(c) Find : $\int\frac{4x+5\,dx}{\sqrt{9-x^2}}$

(d) (i) Express $143\cos\theta - 24\sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$ . Give $\alpha$ correct to four decimal places.

(ii) Solve the equation $143\cos\theta - 24\sin\theta = 100$, for $0<\theta<2\pi$ , giving answer correct to four decimal places.

## Question 3 : START A NEW PAGE

(a) Find the sum of : $3^n + 3^{n-1} + \dots\dots\dots + 3^{-2n}$

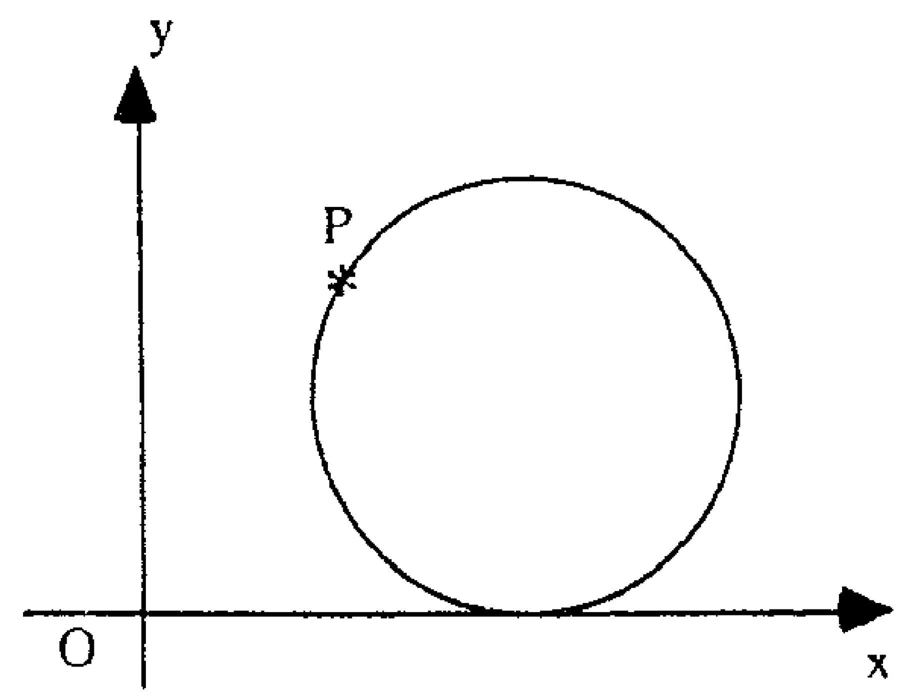
(b) A person borrows \$ 90, 000 at an interest rate of $7\frac{1}{2}$% per annum compounded monthly. It is to be paid with equal monthly instalments over 15 years. Find:

(i) the value of each monthly repayment.

(ii) the amount remaining on the loan after the 120th payment.

(iii) how many repayments are needed if the monthly instalment is increased to \$ 1000 after the 120th payment.

## Question 4 : START A NEW PAGE




The cycloid curve is described by the locus of a fixed point P on the circumference of a circle as the circle moves horizontally along the x-axis . The parametric equations of a cycloid are given by :

$x(\theta) = \theta - \sin\theta$ and $y(\theta) = 1 - \cos\theta$ .

(i) Copy and fill in the following table .

| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| x | | | | | |
| y | | | | | |

(ii) Show that the y–axis is tangent to the cycloid at $\theta = 0$ .

(iii) On a set of x and y axes, graph from the above table the path of P for $0\le\theta\le 2\pi$ .

(iv) The area bounded by the path of P for $0\le\theta\le 2\pi$ and the x-axis is given by :

$$\text{Area} = \int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

Evaluate the exact area.

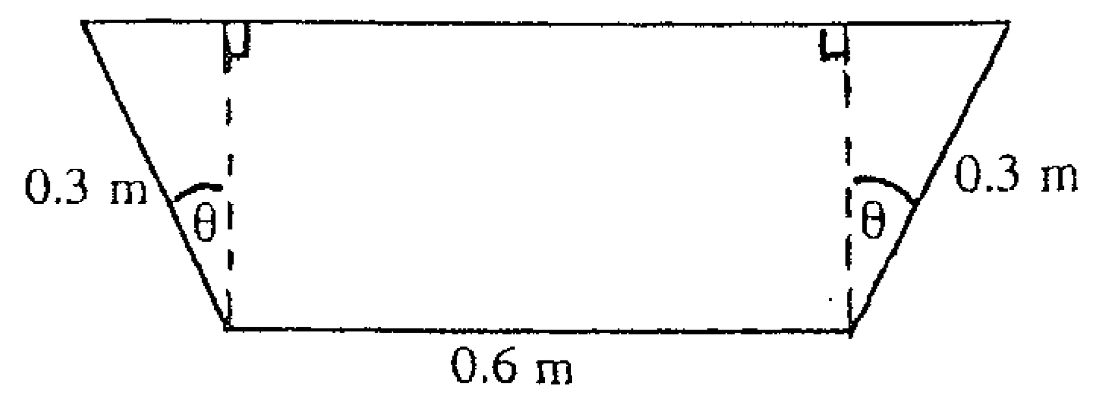
## Question 5: START A NEW PAGE

(a) (i) Graph, on the same axes: $y = \sec x$ and $y = \tan x$ for $0 \le x < \frac{\pi}{2}$ .

(ii) The area bounded by the positive y-axis, the curves $y = \sec x$ and $y = \tan x$, and the line $x = N$ , where $0 < N < \frac{\pi}{2}$, is rotated about the x-axis .
Find the volume of revolution in terms of N .

(iii) Find the volume as N approaches $\frac{\pi}{2}$ .

(b)



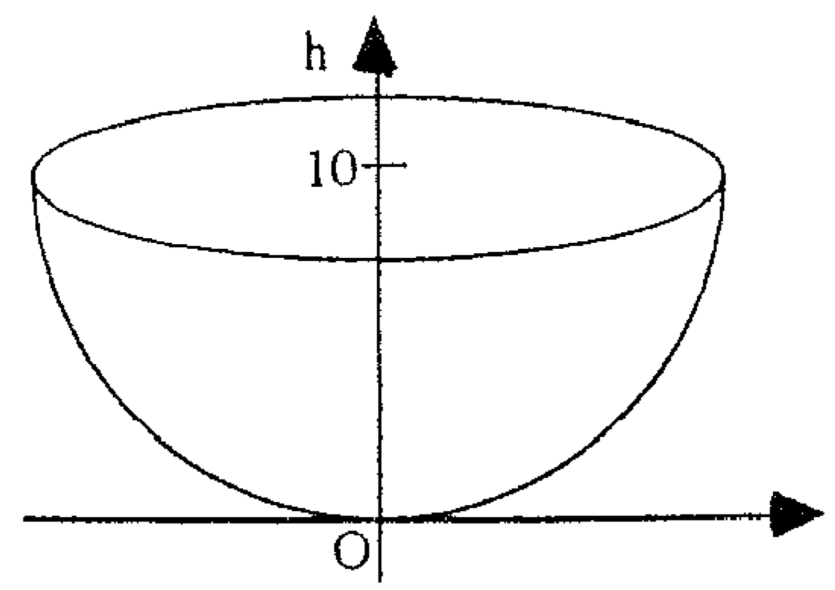

A farmer is to make a discharge chute with a trapezoidal cross-section as shown.

(i) Show the cross-sectional area A is given by : $A = 0.09 \cos\theta \ (2 + \sin\theta) \ m^2$

(ii) Hence find the angle $\theta$ ( to the nearest degree for $0 < \theta < 90^0$ ) which will maximise the cross-sectional area A .

## Question 6 : START A NEW PAGE

(a)




An empty hemi-spherical bowl 20 cm. in diameter rests on a flat surface. Water is poured into the bowl at a constant rate of $20 \ cm^3$ / minute . After t minutes, the water level is h cm. above the base of the bowl. Find the rate at which the water level is rising when the deepest water in the bowl is 6 cm . [You may use $V = \frac{1}{3}\pi h^2 (30-h)$]

(b) (i) Draw a neat sketch of $y = f(x)$, where $f(x) = 2\sin^{-1}x$ .

(ii) On the same axes, graph and **clearly label** $y = f^{-1}(x)$, the inverse of $y = f(x)$.

(iii) Find the exact value of the area bounded by the curves $y = f(x)$ and $y = f^{-1}(x)$, and the line $x = 1$.

**THIS IS THE END OF THE PAPER**

### STANDARD INTEGRALS

$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$

$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$

$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$

$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$

$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

1.

(a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (1)

(b) $\lim_{x \to 0} \frac{\sin 4x}{2x} = \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot 2$ (2)

$= 2$

(c) (i) $\frac{d}{dx} 2\tan x^3 = 6x^2 \sec^2(x^3)$ (2)

(ii) $\frac{d}{dx} x \sin^{-1} x = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$

(d) $y = \operatorname{cosec} x$

$y' = -\operatorname{cosec} x \cot x$

$x = \frac{7\pi}{6}$ $\quad m = -\operatorname{cosec}\frac{7\pi}{6} \cot\frac{7\pi}{6}$ (2)

$= -(-2) \cdot \sqrt{3}$

$= 2\sqrt{3}$

e) $\frac{d}{dx} \cos^{-1}\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1 - \frac{1}{x^2}}} \cdot \frac{-1}{x^2}$

$= \frac{1}{\frac{\sqrt{x^2-1}}{\sqrt{x^2}}} \cdot \frac{1}{x^2}$

$= \frac{1}{|x|\sqrt{x^2-1}}$ (3)

2. (a) $\int_{-4}^{4} \frac{6\,dx}{x^2+16} = 6 \cdot \frac{1}{4}\left[\tan^{-1}\frac{x}{4}\right]_{-4}^{4}$

$= \frac{3}{2}\left\{\tan^{-1} 1 - \tan^{-1}(-1)\right\}$

$= \frac{3}{2}\left\{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right\}$

$= \frac{3\pi}{4}$ (2)

b) $\int \frac{2\,dx}{\sqrt{3-4x^2}} = \int \frac{2\,dx}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}$

$= \sin^{-1}\frac{2x}{\sqrt{3}} + C$ (2)

c) $\int \frac{\ldots}{\sqrt{9-x^2}} \ldots = \int \left(4x(9-x^2)^{\ldots} + \frac{5}{\sqrt{9-x^2}}\right) dx$

$= -4\sqrt{9-x^2} + 5\sin^{-1}\frac{x}{3} + C$ (3)

d(i) $143\cos\theta - 24\sin\theta = R\cos(\theta+\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\therefore$ $R\sin\alpha = 24$ $\quad R > 0 \quad 0 < \alpha < \frac{\pi}{2}$

$R\cos\alpha = 143$

$R = \sqrt{24^2 + 143^2}$

$= 145$

$\tan\alpha = \frac{24}{143}$

$\alpha = 0.1663$

$\therefore 143\cos\theta - 24\sin\theta = 145\cos(\theta + 0.1663)$ (2)

ii) $143\cos\theta - 24\sin\theta = 100$

$145\cos(\theta + 0.1663) = 100$

$\cos(\theta + 0.1663) = \frac{100}{145}$

$\theta + 0.1663 = 0.8098, \; 5.4734$ (3)

$\theta = 0.6435, \; 5.3071$ $\quad 0 < \theta < 2\pi$

3 (a) $S_n = 3^n + 3^{n-1} + \ldots + 3^{-2n}$

$\frac{S_n}{3} = 3^{n-1} + 3^{n-2} \ldots + 3^{-2n} + 3^{-2n-1}$

$\therefore \frac{2}{3} S_n = 3^n - 3^{-2n-1}$ (3)

$S_n = \frac{1}{2}\left[3^{n+1} - 3^{-2n}\right]$

3(b) Principal $P = \$90000$ Interest Rate $= 7\frac{1}{2}\%$ pa

$= 0.625\%$ per month

i) Let $R$ be repayment per month

$\therefore$ Amount owing after 1st Payment $= P(1 + 0.00625) - R$

$A_1 = 1.00625P - R$

Amount owing after 2nd Payment $= (1.00625P - R)1.00625 - R$

$A_2 = 1.00625^2 P - R[1 + 1.00625]$

Amount owing after 3rd Payment $= (1.00625^2 P - R[1 + 1.00625])1.00625 - R$

$A_3 = 1.00625^3 P - R[1 + 1.00625 + 1.00625^2]$

Amount owing after $n$ Payments $= 1.00625^n P - R[1 + 1.00625 + \dots 1.00625^n]$

$$A_n = 1.00625^n . P - R\left[\frac{1.00625^n - 1}{0.00625}\right] \quad -(*)$$

For 15 years $n = 180$ $A_{180} = 0$ $P = 90000$

$$\therefore R\left[\frac{1.00625^{180} - 1}{0.00625}\right] = 90000 . 1.00625^{180}$$

$$R = \frac{90000 . 1.00625^{180} . 0.00625}{1.00625^{180} - 1}$$

(4)

Monthly Repayment $= \$834.31$

ii) $A_{120} = 90000 . 1.00625^{120} - 834.31\left(\frac{1.00625^{120} - 1}{0.00625}\right)$

Amount owing end 120th Payment $= \$41636.75$ (2)

iii) $R = 1000$ $P = 41636.75$, $A = 0$. by (*)

$$1.00625^n . 41636.75 = 1000\frac{(1.00625^n - 1)}{0.00625}$$

$= 160000 . 1.00625^n - 160000$

$$1.00625^n = \frac{160000}{118363.25}$$

$= 1.35$ (3)

$$n = \frac{\ln 1.35}{\ln 1.00625} = 48.38 \Longrightarrow 49 \text{ payment}$$

4(i)

| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $x$ | $0$ | $\frac{\pi}{2} - 1$ | $\pi$ | $\frac{3\pi}{2} + 1$ | $2\pi$ |
| $y$ | $0$ | $1$ | $2$ | $1$ | $0$ |

(3)

(ii) $y = 1 - \cos\theta$ $\quad$ $x = \theta - \sin\theta$

$\frac{dy}{d\theta} = \sin\theta$ $\quad$ $\frac{dx}{d\theta} = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} \quad (1)$$

At $\theta = 0$ $\quad \frac{dy}{dx} = \lim_{\theta \to 0}\left(\frac{\sin\theta}{1 - \cos\theta} . \frac{1 + \cos\theta}{1 + \cos\theta}\right)$

$$= \lim_{\theta \to 0} \frac{\sin\theta . (1 + \cos\theta)}{1 - \cos^2\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta(1 + \cos\theta)}{\sin^2\theta}$$

$$= \lim_{\theta \to 0}\left(\frac{1 + \cos\theta}{\sin\theta}\right)$$

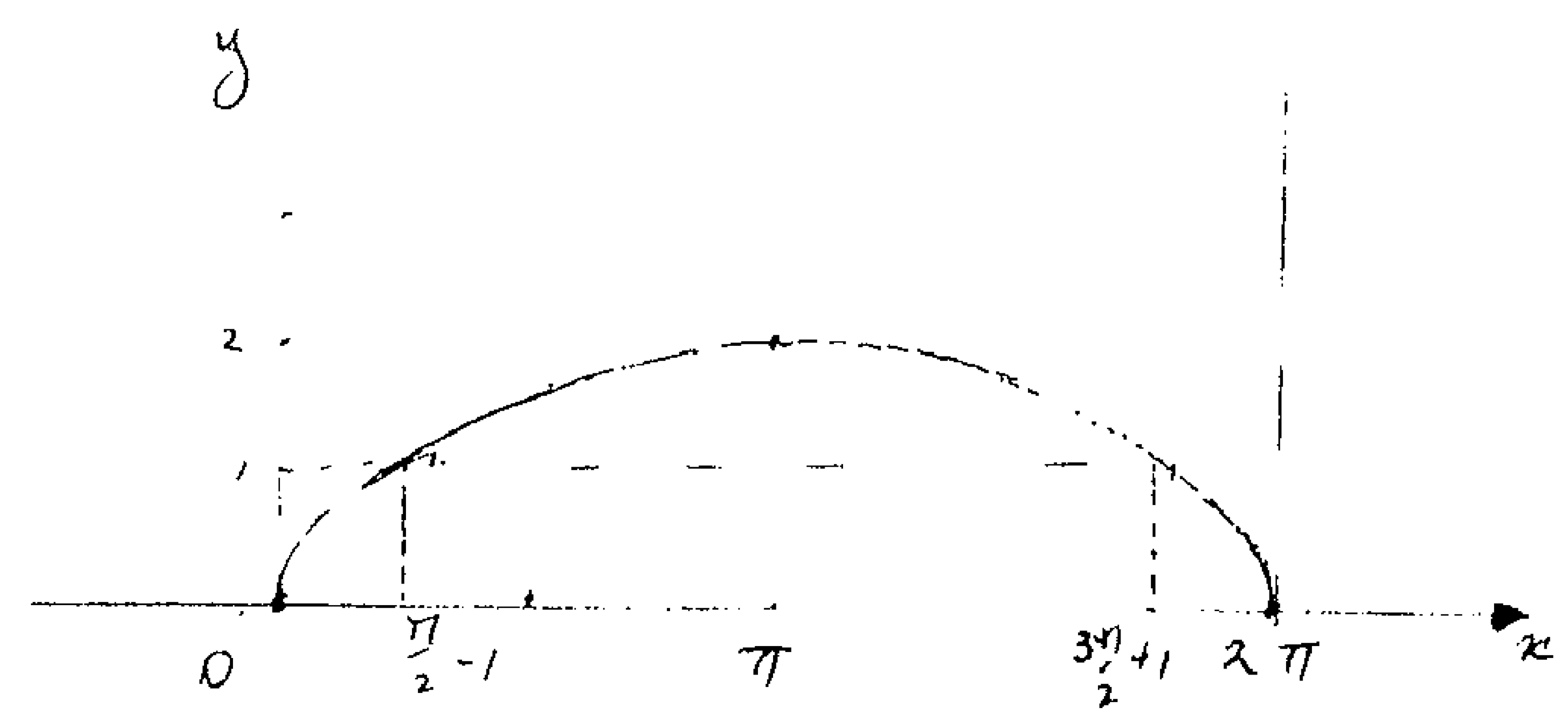
$= \infty$.

ie gradient is vertical when $\theta = 0$ (1)

ie Equation Tangent is $x = 0$

$\therefore$ $y$ axis is a tangent to the cycloid

iii)




(2)

(iv) $A = \int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$

$= \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$

$= \int_0^{2\pi} 1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2} \, d\theta$

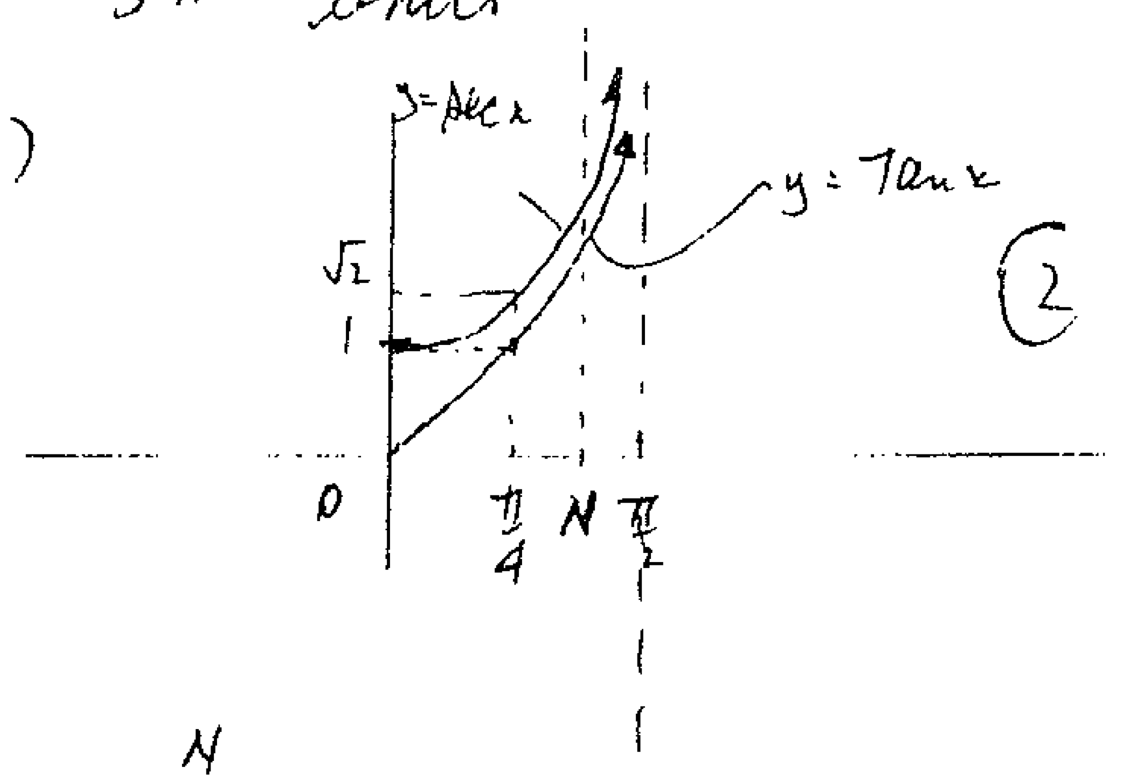
$= \int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta \, d\theta$

$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= 3\pi - 0 + 0 - 0$ (5)

$= 3\pi$ units$^2$

5 (a) (i)




(2)

(ii) $V = \pi \int_0^N (y_1^2 - y_2^2)\,dx$

$= \pi \int_0^N \sec^2 x - \tan^2 x \, dx$

$= \pi \int_0^N 1 . \, dx$

$= \pi [x]_0^N$ (2)

$= \pi N$ units$^3$

(iii) $\lim_{N \to \frac{\pi}{2}} V = \frac{\pi^2}{2}$ units$^3$ (1)

5(b) Area Trapezium $= \frac{h}{2}(x+y)$

$= \frac{1}{2} \cdot 0.3\cos\theta \left[0.6 + 0.6 + 2 \times 0.3 \sin\theta\right]$

(7)

$A = 0.09 \cos\theta \left[2 + \sin\theta\right]$

$A' = 0.09 \left\{-\sin\theta(2+\sin\theta) + \cos^2\theta\right\}$

$= 0.09 \left\{\cos^2\theta - \sin^2\theta - 2\sin\theta\right\}$

$= 0.09 \left\{\cos 2\theta - 2\sin\theta\right\}$

$A'' = 0.09 \left\{-2\sin 2\theta - 2\cos\theta\right\}$

For maximum area

$A' = 0$

$\cos^2\theta - \sin^2\theta - 2\sin\theta = 0$

$1 - 2\sin^2\theta - 2\sin\theta = 0$

$2\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \frac{-2 \pm \sqrt{4+8}}{4}$

$= \frac{\sqrt{3}-1}{2}$ only $\sin\theta > 0$ $0 < \theta < 90°$

$\theta = 0.37$ rad

$\theta = 21°$ (nearest degree)

we will test $A''$ for sign concavity, since $A, A', A''$ are continuous in domain $0 < x < \frac{\pi}{2}$

$A''(0.37) = -0.288$

$< 0$

$\therefore$ there is a relative maximum at $\theta = 21°$, but since there is only one turning point in domain $0 < \theta < \frac{\pi}{2}$ there is an absolute maximum cross-sectional area when $\theta = 21°$.

6(a) $V = \frac{1}{3}\pi h^2(30-h)$

$V = \frac{\pi}{3}(30h^2 - h^3)$

$\frac{dV}{dh} = \frac{\pi}{3}(60h - 3h^2)$

$= \pi(20h - h^2)$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\frac{dV}{dt} = \pi(20h - h^2)\frac{dh}{dt}$

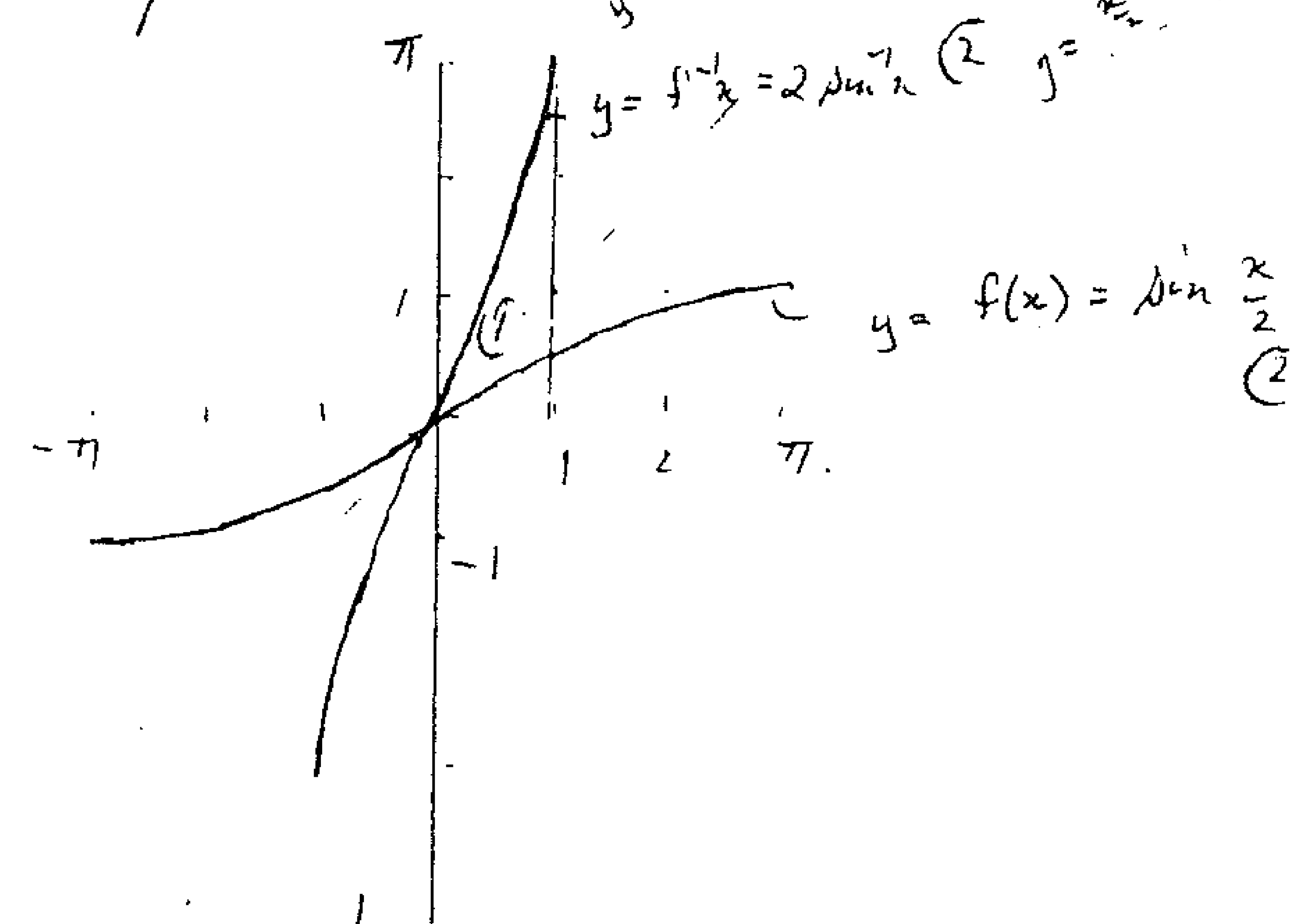
$h = 6\,cm \quad \frac{dV}{dt} = 20\ cm^3/minute$

$20 = \pi(20\times 6 - 6^2)\cdot\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{20}{84\pi}$ (3)

rate rise $= \frac{5}{21\pi}$ cm/minute

b(i)




$A = \int_0^1 2\sin^{-1}x\,dx - \int_0^1 \sin\frac{x}{2}\,dx$

$= \pi - \int_0^{\pi} \sin\frac{x}{2}\,dx - \int_0^1 \sin\frac{x}{2}\,dx$

$= \pi + \left[2\cos\frac{x}{2}\right]_0^{\pi} + \left[2\cos\frac{x}{2}\right]_0^1$ (4)

$= \pi + 0 - 2 + 2\cos\frac{1}{2} - 2$

$= \pi + 2\cos\frac{1}{2} - 4$ units$^2$